Models of Set Theory II - Winter 2015/2016

Prof. Dr. Peter Koepke, Regula Krapf Problem sheet 10

Problem 37 (6 points)**.** Suppose that $\mathbb{P}$ is a σ-closed forcing.

(a) Let κ be an uncountable cardinal in M and suppose that $p \Vdash_{\mathbb{P}}^{M}$ "$\dot{C} \subseteq [\check{\kappa}]^{\check{\omega}}$ is club" for some $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{C}$. Prove that there is a club $D \subseteq [\kappa]^{\omega}$ in M such that for all $x \in D$ there are sequences $\langle p_n \mid n \in \omega \rangle$ and $\langle x_n \mid n \in \omega \rangle$ such that for all $n \in \omega$, $p_{n+1} \leq_{\mathbb{P}} p_n \leq_{\mathbb{P}} p$, $x_{n+1} \supseteq x_n$, $p_n \Vdash_{\mathbb{P}}^{M} \check{x}_n \in \dot{C}$ and $\bigcup_{n\in\omega} x_n = x$.

(b) Conclude that $\mathbb{P}$ is proper.

Problem 38 (4 points)**.** The *Proper Forcing Axiom (PFA)* states that for every proper forcing $\mathbb{P}$, $\mathrm{FA}_{\aleph_1}(\mathbb{P})$ holds. Its consistency proof uses strong large cardinal assumptions. Show that the if we replace $\aleph_1$ by $\aleph_2$ above, the statement becomes false.

Hint: Consider a forcing which collapses $\aleph_2$ to $\aleph_1$.

Problem 39 (6 points)**.** Let $\mathbb{P} = {}^{<\omega_1}\omega$, ordered by reverse inclusion. We construct a countable support iteration of length ω of subforcings of $\mathbb{P}$. We define inductively $\mathbb{P}_n$-names $\dot{\mathbb{Q}}_n$ and $\dot{f}_n$ with the following properties:

- Let $\dot{\mathbb{Q}}_0 = \check{\mathbb{P}}$.
- $\mathrm{dom}(\dot{\mathbb{Q}}_n) \subseteq \{\check{p} \mid p \in \mathbb{P}\}$ and $\mathbb{1}_{\mathbb{P}_n} \Vdash_{\mathbb{P}_n}^{M}$ "$\dot{\mathbb{Q}}_n$ is dense in $\check{\mathbb{P}}$".
- $\dot{f}_n$ is the canonical $\mathbb{P}_{n+1}$-name for the generic function $f_n : \omega_1 \to \omega$ added by $\dot{\mathbb{Q}}_n$.
- Given $\dot{\mathbb{Q}}_n$, let $\dot{\mathbb{Q}}_{n+1}$ name the set of all $p \in \mathbb{P}$ such that $\mathrm{dom}(p)$ is either $\emptyset$ or $\alpha + \dot{f}_n^{G_{n+1}}(\alpha + \omega)$ for some limit ordinal α.

Prove the following statements:

(a) $\mathbb{P}$ has an antichain of size $\aleph_1$.

(b) If $p \in \mathbb{P}_\omega$ and $n + 1 \in \mathrm{supp}(p)$ then $p \restriction n + 1 \Vdash_{\mathbb{P}_{n+1}}^{M} \mathrm{dom}(p(n + 1)) < \mathrm{dom}(p(n))$.

(c) Each $\mathbb{P}_n$ is σ-closed but $\mathbb{P}_\omega$ collapses $\aleph_1$. *Hint:* Use Problems 16 and 17.

Problem 40 (4 points)**.** Let $\langle \langle \mathbb{P}_\alpha, \leq_\alpha, \mathbb{1}_\alpha \rangle \mid \alpha \leq \kappa \rangle$ denote the countable support iteration of the sequence $\langle \langle \dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha \rangle \mid \alpha < \kappa \rangle$. Assume that each $\dot{\mathbb{Q}}_\alpha$ is a full name and $\mathbb{1}_\alpha \Vdash_{\mathbb{P}_\alpha}^{M}$ "$\dot{\mathbb{Q}}_\alpha$ is σ-closed". Prove that $\mathbb{P}_\kappa$ is σ-closed.

Revision exercises

Problem 41 (6 points)**.** A *Ramsey ultrafilter* is an ultrafilter $\mathcal{U} \subseteq [\omega]^\omega$ which contains all co-finite subsets of ω and such that for every *colouring* $\pi : [\omega]^2 \to 2$ there is $x \in \mathcal{U}$ such that $\pi \restriction [x]^2$ is constant. For $x, y \in [\omega]^\omega$ we say that x is *almost included* in y, denoted $x \subseteq^* y$, if $x \setminus y$ is finite. Now let $\mathbb{U}$ denote the forcing $\langle [\omega]^\omega, \subseteq^*, \omega \rangle$. Prove the following statements:

(a) $\mathbb{U}$ is σ-closed. *Hint:* Use some previous exercise on Sheet 6.

(b) Show that if G is M-generic for $\mathbb{U}$ then G is a Ramsey ultrafilter.

Problem 42 (8 points)**.** Let $\mathfrak{m}$ denote the least cardinal κ such that MA_κ fails. Let μ denote the Lebesgue measure on $\mathbb{R}$. For $\varepsilon > 0$ let $\mathbb{P}_\varepsilon$ be the forcing whose conditions are open sets $p \subseteq \mathbb{R}$ with $\mu(p) < \varepsilon$, ordered by reverse inclusion.

(a) Let $\{p_\xi \mid \xi < \omega_1\} \subseteq \mathbb{P}_\varepsilon$ and $\delta > 0$ such that $\mu(p_\xi) < \varepsilon - 3\delta$ for all $\xi < \omega_1$. Show that there are $\xi, \eta < \omega_1$ with $\xi \neq \eta$ such that p_ξ and p_η are compatible. Conclude that $\mathbb{P}_\varepsilon$ is ccc.

(b) Show that if G is M-generic for $\mathbb{P}_\varepsilon$ then $\mu(\bigcup G) \leq \varepsilon$.

(c) Let $N \subseteq \mathbb{R}$ with $\mu(N) = 0$. Show that $D_N = \{p \in \mathbb{P}_\varepsilon \mid p \supseteq N\}$ is dense.

(d) Show that $\mathfrak{m} \leq \mathrm{add}(\mathcal{N})$.

Hint for (a),(b): Approximate conditions using unions of rational intervals.

Problem 43 (4 points)**.** Let $\mathbb{P}$ be κ-distributive and $\mathbb{1}_\mathbb{P} \Vdash^M_\mathbb{P}$ "$\dot{\mathbb{Q}}$ is $\check{\kappa}$-distributive". Prove that $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-distributive.

Problem 44 (8 points)**.** Suppose that $\mathbb{M} = \langle M, \mathcal{C} \rangle \models \mathsf{GBC}$. We say that a class forcing $\mathbb{P}$ satisfies the

- Ord-cc, if every antichain of $\mathbb{P}$ is set-sized (i.e. an element of M).
- *maximality principle*, if for every first-order formula $\varphi(x)$ (possibly with class name parameters) and for every $p \in \mathbb{P}$, if $p \Vdash^\mathbb{M}_\mathbb{P} \exists x \varphi(x)$ there is $\sigma \in M^\mathbb{P}$ such that $p \Vdash^\mathbb{M}_\mathbb{P} \varphi(\sigma)$.

Prove the following statements:

(a) If $\sigma \in M^\mathbb{P}$ and G is $\mathbb{M}$-generic for $\mathbb{P}$ then $\mathrm{rank}(\sigma^G) \leq \mathrm{rank}(\sigma)$.

(b) $\mathbb{P}$ satisfies the Ord-cc if and only if it satisfies the maximality principle.

(c) Conclude that the maximality principle can fail for class forcings.

Please hand in your solutions on Monday, 01.02.2015 before the lecture.